WELL-POSEDNESS AND STABILITY OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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in Partial Fulfilment of the Requirements
for the Degree of

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Certificate

Certified that this work on **Stability and Well-posedness of linear partial differential equations** by Mr Anand Kumar has been carried out under my supervision and that this work has not been submitted elsewhere for a degree

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Abstract

The problem of closed loop system stability is studied in time domain when input and outputs are in *Schwartz space* and at least continuous in time The system, being studied, consists of a plant and a controller in feedback interconnection, which are modelled by *partial differential operators* of a particular class

An attempt is also made to formulate the *Stabilization problem* in a more general framework by giving a new definition of stability, termed as *Topological Stability*, when input and outputs are in arbitrary *Topological vector space* It has been shown that *BIBO stability* and *V-Stability* are special cases of *Topological Stability*

Chapter 1

Introduction

In feedback control theory, *Stability* of the feedback system is of vital importance Primarily the notion of stability considered is the **Lyapunov** stability of an equilibrium point of the dynamical system represented by feedback system This is also called *internal stability* The convergence of state for future time is taken as the basic requirement of *internal stability* However, most engineering systems are also excited by external inputs in addition to the initial condition For instance, the noise and the reference inputs to a feedback system **Lyapunov** stability is thus not adequate for studying such general dynamical systems. For this reason the notion of *input-output stability* is more relevant for engineering systems

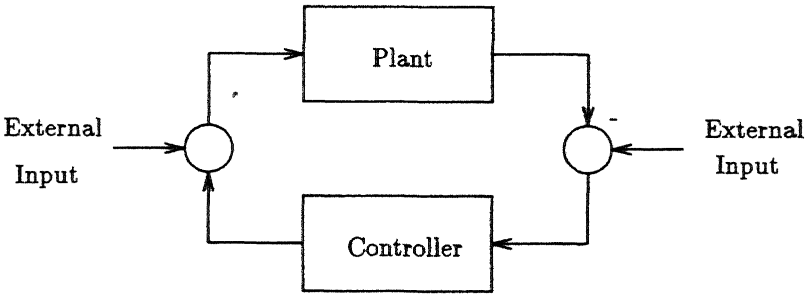

Figure 1 1 Feedback system

Systems excited by inputs are often considered as operators between normed spaces Boundedness or continuity of such an operator is physically relevant in engineering [26]. An operator is considered *input-output stable* if for a bounded input the output of the system is also bounded [29] This notion of stability is particularly useful for linear systems represented by convolution operators. Then the *input-output stability* of such systems depends on the notion of boundedness such as L_1 L_2 L_∞ etc

When the system is governed by partial differential equations(PDE) inputs and outputs are not necessarily normed spaces Hence the notion of boundedness of inputs and outputs as considered above changes Consequently the conditions for stability and the problem of stabilization by feedback needs to be revised Well known examples of such systems are transmission lines vibrating string temperature distribution over a rod heated at one end etc An objective of this thesis is to develop a notion of input output stability over *topological vector space* and to investigate the stabilization problem for operators arising from a class of PDE s

1 1 Input-output Stability and the Stabilization problem

To begin we first describe the standard notion of the *input output stability* and the problem of stabilization in feedback systems For the sake of clarity we only consider the space $L_2[0\ \infty)$ as the space of bounded inputs and the space $C[0\ \infty)$ as the space of all inputs Let P U $\mapsto$ U denote a linear operator called the plant

Definition 1 1 1 The linear operator P U $\mapsto$ U is called L_2 stable(BIBO stable) if for every input $u \in L_2[0\ \infty) \cap C[0,\infty)$ the output $y = \mathrm{P}(u)$ also belongs to $L_2[0,\infty) \cap C[0,\infty)$

For a given plant operator P U $\mapsto$ U and another operator C U $\mapsto$ U called the controller, the feedback system is described by the following equations

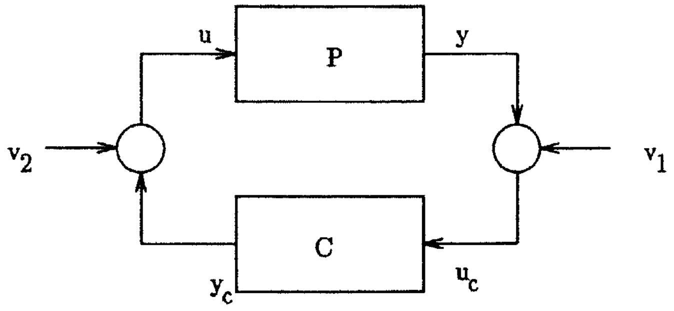

Figure 1 2 Closed loop system

$$\left.\begin{array}{rcll} y &=& \mathrm{P}u & \textit{Plant} \\ y_c &=& \mathrm{C}u_c & \textit{Controller} \\ \left.\begin{array}{rcl} u_c &=& v_1 - y \\ u &=& y_c + v_2 \end{array}\right] & & & \textit{Feedback Interconnection} \end{array}\right] \qquad (1\ 1)$$

Here v_1 and v_2 denote external inputs

1 1 1 The stabilization problem

The interconnection of plant and the controller operator as a feedback system described by equations 1 1 is diagrammatically shown in fig 1 1 From equations 1 1 the pair y y_c can be eliminated to get the equations of the closed loop system as

$$\left.\begin{array}{rcrcl} Pu & + & u & = & v_1 \\ u & - & Cu_c & = & v_2 \end{array}\right] \qquad (1\ 2)$$

Definition 1 1 2 [L_2–Stabilization problem] Given the plant operator P determine the existence and characterization of all controllers C such that the following conditions are satisfied

1 The feedback system determines a unique operator T $U \times U \mapsto U \times U$ mapping internal inputs $(u\ u_c)$ in $U \times U$ uniquely to external inputs $(v_1\ v_2)$ in $U \times U$ When the equations 1 2 have a unique solution $(u\ u_c)$ in $U \times U$ for every pair of inputs $(v_1\ v_2)$ in $U \times U$ then the feedback system is said to be well posed

2 The operator T satisfies

$$T(L_2 \times L_2) \subseteq L_2 \times L_2$$

The stabilization problem thus clearly depends on the notion of stability considered for linear operators T as well as on the subspace of the bounded inputs and outputs of U For example if the stable linear operators on a banach space B are considered as the class of continuous linear maps on B then the *stabilization problem* has conditions a) U=B and well–posedness and b) T is continuous on $B \times B$

In the statement of *stabilization problem* in the definition given above stability is considered equivalent to having $L_2[0\ \infty)$ as an invariant subspace Hence the above version of the *stabilization problem* may be called as an *invariant subspace theoretic problem* of stabilization

1 2 Previous work and Motivation

The notion of *bounded input bounded output*(BIBO) was developed by **Zames** [29] and **Williams** [28] primarily for linear operators and condition for closed loop stability were developed by extending the Nyquist criterion [15] **Willems** observed that the *well posedness* of feedback system was very crucial in order to define the closed loop stability [26] [27] Various important cases of linear operators and L_p boundedness were analyzed in [7] [6] [8] [9] and [25] Particularly frequency domain ideas began to reappear in these works (after Nyquist) for the problems of linear feedback stabilization problem

The complete algebraic conditions for closed loop stability for linear multivariable (matrix valued operators) systems was developed by **Desoer** and **Wang** in terms of transfer functions Almost simultaneously system described by transfer functions the problem was solved completely using algebraic methods by **Vidyasagar** [20] The *stabilization problem* in the form defined above became clear only after these works It s application to feedback system design were developed in [24] This entire theory is known as *factorization theory* of control systems

The *factorization theory* has not been found convenient for studying the *stabilization problem* for systems governed by partial differential equations Nevertheless a convolution algebra of particular distributions is developed by **Callier** and **Desoer** in [2] which can represent BIBO stable systems governed by PDE s wherein the algebraic theory of stabilization is extended to this case It is however not clear how well this algebra can model the convolution systems obtained from general class of PDE s In fact it is well known that the PDE s have deep analytical properties and their purely algebraic description may not be very fruitful It seems therefore, more appropriate to analyze the *stabilization problem* for special case of PDE s for which the spaces are known wherein the solution exists

The original Nyquist criterion for the stability of a feedback system provides only a condition for stability when the operator C is known Hence it may not be useful for developing a characterization (or existence proof) of C for a given plant operator P However for the cases of operators arising from PDE s, such a condition for stability might be all that can be obtained It is thus worthwhile to develop a Nyquist criterion for feedback systems with more general operators and the notion of stability On these lines an *index theoretic criterion* for stability was developed for convolution systems in [4] Further extension of this theory to *Fredholm operators* is essentially an open problem

1 3 Organization of thesis

The notion of topological stability motivated by partial differential equations is introduced in chapter 2 Also it is shown that it is defined on the lines of BIBO stability A condition is also obtained which will render closed loop feedback system stable for a particular class of linear partial differential operators

In chapter 3 the topological stabilization is formulated and it has been shown using first order plant that we can give a class of controllers using the condition obtained in chapter 2 which will stabilize the feedback system

Scope for future work in this area is presented in chapter 3 with possible approaches to attack this problem

A quick review of mathematical background is presented in appendixes

1 4 List of symbols

Following is the list of some symbols with their standard meaning

1 $C_0^\infty(X)$ space of C^∞ functions with compact support in $X \subseteq R^n$

2 $\sigma(A)$ set of eigenvalues of A, also called the point spectrum of A

3 $\mathcal{S}(R^n)$ schwartz space of functions on R^n(see Appendix A)

4 T_1 T_2 denotes topological vector spaces

5 V_{O1}, V_{O2} denotes corresponding subspaces of T_1 and T_2

6 $\mathcal{D}'(R^n)$ space of distributions on R^n

7 $\mathcal{S}'(R^n)$ space of tempered distributions on R^n

8 $C_t^m(T)$ space of m times continuously differentiable functions which map a open subset of positive real line to space T

See Appendix A for further notations

Chapter 2

Input-Output stability over Topological spaces

In this chapter we describe various notions of stability and define a more comprehensive notion of stabilitv over topological vector spaces which is motivated by partial differential operators(PDO) Subsequently the stabilization problem for particular class of PDE s is vigorously defined and a sufficiency condition is obtained

2 1 General definition of stability

In literature systems are discussed with the constraint that input and output spaces are normed spaces [24] But we will be discussing a more general version of it when input and output are *topological vector spaces* Let T_1 and T_2 be two *topological vector spaces* and $S \overset{\mathrm{d\,f}}{=} [t \quad \infty)$ Let us consider $V_{O1} \subseteq T_1$ and $V_{O2} \subseteq T_2$ respectively be the spaces of stable inputs and stable outputs which are given The feedback system is shown in fig 1 2 $u \quad S \mapsto T_1$ and $y \quad S \mapsto T_2$ are respectively the input and the output of the Plant system represented by operator P $T_1 \mapsto T_2$ Similarly $u_c \quad S \mapsto T_2$ and $y_c \quad S \mapsto T_1$ are the input and the output of controller system represented by C $T_2 \mapsto T_1$ All maps from S to T_i, $i = 1, 2$ are continuous Let $I_1 \quad T_1 \mapsto T_1$ and $I_2 \quad T_2 \mapsto T_2$ such that $I_i(x) = x \ \forall \ x \in T_i$ The feedback system represented in equation 1 2 can also be represented as follows

$$\begin{pmatrix} P & I_2 \\ I_1 & -C \end{pmatrix} \begin{pmatrix} u \\ u_c \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2 1}$$

Therefore the closed loop operator

$$T_c = \begin{pmatrix} P & I_2 \\ I_1 & -C \end{pmatrix} \quad T_1 \times T_2 \mapsto T_2 \times T_1 \tag{2 2}$$

2 1 1 Topological stability of a system

In this subsection we describe the general notion of stability for a system shown in figure 2 1

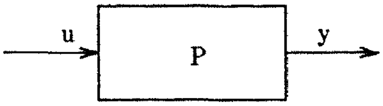

Figure 2 1 System Stability

Definition 2 1 1 [Topological stability] A system represented by $P\ T_1 \mapsto T_2$ is topologically stable if

1 $\mathrm{Image}(P) \supseteq V_{O2}$

2 P^{-1} is one one and continuous

3 $P(V_{O1}) \subseteq V_{O2}$

This notion of system stability can be extended to feedback systems shown in figure 1 2

Definition 2 1 2 [Topological stability of feedback systems] A feedback system $T_c\ \ T_1 \times T_2 \mapsto T_2 \times T_1$ is topologically stable if

1 T_c is one one and continuous

2 $\mathrm{Image}(T_c^{-1}) \supseteq V_{O1} \times V_{O2}$

3 $T_c^{-1}(V_{O2} \times V_{O1}) \subseteq V_{O1} \times V_{O2}$

It is important to note that T_c maps internal inputs $(u, u\ )$ to external inputs $(v_1\ v_2)$ whereas open loop system maps input u to output y

This new notion of stability is not claimed to be superior than BIBO stability and V Stability The future prospects of this notion in Control Theory are to be seen However it is shown here that topological stability is equivalent to BIBO stability and V Stability when topological spaces of inputs and outputs are suitably structured

2 2 V-Stability

This notion of stability was developed by **Vidyasagar** [21] In this subsection we will show that V Stability is a special case of topological stability when topological spaces are suitably structured Let X and Y be Banach spaces and D is dense in X Let there be a linear operator A D $\mapsto$ Y

Definition 2 2 1 [V Stability] System represented by A is V Stable if there exists linear extension A of A such that Domain(A)=X and Range(A)=Y

A linear operator which is BIBO stable is automatically V Stable but not vice versa Examples are systems represented by linear unbounded operators

Proposition 2 2 1 Let T_1 and T_2 be Banach spaces and if V_{O1} is dense in T_1 and also V_{O2} is same as T_2, then a linear operator P $T_1 \mapsto T_2$ is V Stable

Proof Image(P) $\supseteq V_{O2} = T_2$ implies that Image(P)=T_2 Hence P is onto T_2 P^{-1} is one one $\Rightarrow$ that if $P^{-1}(y_1) = P^{-1}(y_2) = x$ then $y_1 = y_2$(by definition) P^{-1} one one does not rule out the possibility that $P^{-1}(y) = x_1$ and $P^{-1}(y) = x_2$ which can be written as $P(x_1) = y$ and $P(x_2) = y$ As P is linear $P(x_1 + x_2)=P(x_1)+P(x_2)=2y=2P(x_1)=P(2x_1)$ which implies that $x_1 = x_2$ Therefore P is one one and onto As T_1 and T_2 are closed and P is one-one and onto hence P is continuous by Open mapping theorem Hence P is V Stable □

2 3 Bounded input bounded output stability

Let $T_1 = T_2 = (N \ \| \ \|_N)$ a normed space We define a Truncation operator L_T

$$L_T x(t) = \begin{cases} x(t) & t_o \leq t \leq T \\ 0 & T < t \end{cases} \qquad x(t) \in N \tag{2 3}$$

Let $\| \ \|$ be the norm on the space of functions $x(t)$ $S \mapsto N$ Let us define N_e, an extended normed space as follows N_e is the set of all functions $x(t) \in N$ such that

1 $\forall x(t) \in (N)_e$, $T \mapsto \|L_T[x(t)]\|$ is monotonically increasing function

2 $\forall x(t) \in N_e$, $\|L_T[x(t)]\| \to \|x\|_N$ as $T \to \infty$

Definition 2 3 1 [BIBO Stability] A linear system P $N \mapsto N$ is BIBO stable if $P(N_e) \subseteq N_e$

If $V_{O1} = V_{O2} = N_e$ then with above assumptions topological stability implies BIBO stability

2 4 Topological stability and Index

In this section we relate topological stability with index of a Fredholm kind of operator for a special case

Definition 2 4 1 [Fredholm Operator] An operator is called a Fredholm operator if the kernal and cokernal spaces of the operator are finite dimensional

Fredholm operators are also called as Noether operators

Example 2 4 1 Let us consider a scalar linear time invariant system with unity feedback Here $T_1 = T_2 = L_2[0\ \infty)$ and $V_{O1} = V_{O2} = L_e$ From the figure we can write following

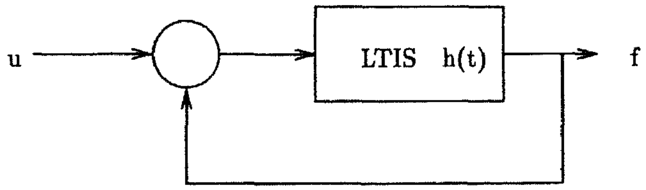

Figure 2 2 Unity feedback system

equations governing the system

$$\begin{aligned} (u-f)*h &= f && (2\ 4)\\ u*h-f*h &= f && (2\ 5)\\ f(t)+\int_0^\infty h(t-\tau)\,f(\tau)\,d\tau &= \int_0^\infty h(t-\tau)\,u(\tau)\,d\tau && (2\ 6) \end{aligned}$$

If $u(t) \in L_e$ then under what conditions $f(t) \in L_e$ Solving in transformed domain, it has been shown see [15] [1], that if $|1+h(s)|$ satisfy Nyquist criteria then the closed loop system is stable

This condition can be put using index as number of encirclements of $|1+h(s)|$ is same as the degree of the curve $h(s)$ at 1 Since for polynomials, if corresponding operator in time domain is H, then Index(H) = Deg[$h(s)$] (see [1]) and hence system is stable if Index(H) = 0

Let T_1 and T_2 be finite dimensional topological spaces denoted by T and V_O be the corresponding subspace of stable inputs and outputs

Proposition 2 4 1 **A linear Fredholm operator P $T \mapsto T$ representing a plant is topologically stable if Index($P_{|V_0}$)=0**

The proof is very straight forward

2 5 Well-posedness

The concept of well posedness was first introduced by Hadamard [10] The mathematical model can be considered satisfactory only in the case when for some class of data of the problem the problem has a solution and it is unique However even this is not sufficient Data found using measurements is not perfect and always has an error The problem can be considered well posed only in the case when a small change in data of the problem leads to small change in the solution

This is not true for all problems It is important to take account of the structure of the equation when posing a boundary value problem Moreover in defining well posedness an important role is played by proper choice of function spaces for the solution of the problem under consideration Most general definition of well posedness is given below

Definition 2 5 1 [Well posedness] Let U V,F be topological vector spaces and $U \subset V$ Also u be a vector valued function which satisfies initial value problem (Cauchy problem) and f is a vector valued data of the problem The Cauchy problem is well posed if

1 for each $f \in F$ there exists a solution $u \in U$ of the Cauchy problem being studied

2 the solution is unique

3 the solution u as an element of the space V depends continuously on $f \in F$

Remark In our case an operator is said to be well posed over some space if the corresponding Cauchy Problem is well posed over same space

2 6 Stability over Schwartz spaces

In this section a special case of stabilization problem is studied on Schwartz space Formulation of stabilization problem on topological spaces is given in next chapter Firstly a seminorm is defined on $C_t[\mathcal{S}(R^n)]$ and then we properly formulate the problem In the last subsection we present a result

2 6 1 Seminorm on continuous functions valued in Schwartz Space

On Schwartz space $\mathcal{S}(X)$ we have standard seminorm defined as given in equation A 6 But all input and output of plant and controller are Schwartz space valued maps i e they belong to $C_t[\mathcal{S}(R^n)]$ So in addition to equation A 6 we need to have some additional seminorm type of continuous function which we define below

Definition 2 6 1 A function $f(t\ x) \in C_t[\mathcal{S}(R^n)]$ is said p bounded if

$$\|f\|_{\alpha\,\beta\,p} = \left(\int_S \left(\sup_{x\in R} |x^\alpha \partial^\beta f(t\ x)|\right)^p dt\right)^{1/p} < \infty \quad \forall\, \alpha\ \beta \qquad (2\ 7)$$

Similarly if vector $F(t\ x) \in C_t[\mathcal{S}(R^n)]^n$ then we use any l_q norm on finite dimensional vector Equation 2 8 gives the expression

$$\|F\|_{\alpha\,\beta\,p\,q} = \left(\sum_j (\|f_j\|_{\alpha\,\beta\,p})^q\right)^{1/q} \qquad (2\ 8)$$

where f_j are components of vector $F(t\ x)$

2 6 2 Stabilization problem statement on Schwartz space

The class of linear partial differential operators which will represent plant and controller is the following simple class

$$P(\partial_t\ \partial_x) = \partial_t^m + a_1(\partial_x)\partial_t^{m-1} + \quad + a_m(\partial_x) \qquad (2\ 9)$$

where $\partial_t^m = \frac{1}{i^m}\frac{\partial}{\partial t}$ and $x \in R^n$and $t \in S$ $a_i(\ )$ are C^∞ functions in R^n m is called the order of the operator $T_1 = T_2 = \mathcal{S}(R^n)$ And $v_1\ v_2\ u_c\ f_c\ u, f \in C_t[\mathcal{S}(R^n)]$ $C_t[\mathcal{S}(R^n)]$ is the space

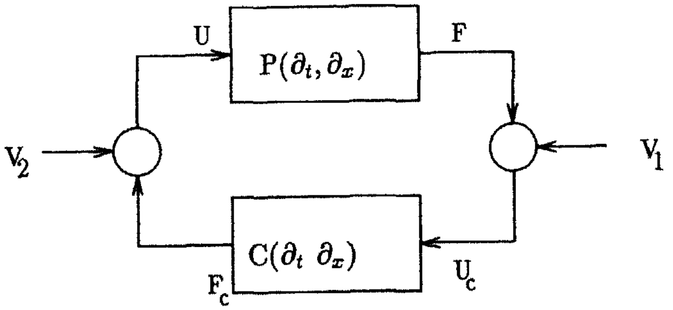

Figure 2 3 Thesis Problem

of at least continuous functions in t and valued in $\mathcal{S}(R^n)$ V_O is the space of functions $z(t,x)$

$\in \mathrm{C}_t[\mathcal{S}(\mathrm{R}^n)]$ such that $\|z(t\ x)\|_{\alpha\ \beta\ p\ q} \to 0$ (see equation 2 8) as $t \to \infty$ V$_{O1}$ and V$_{O2}$ are V$_O$ So given P($\partial_t\ \partial_x$) find C($\partial_t\ \partial_x$) such that closed loop system is BIBO stable assuming all initial conditions zero i e

$$\forall v_1\ v_2 \in \mathrm{C}_t[\mathcal{S}(\mathrm{R}^n)] \to \mathrm{u}\ \mathrm{u_c} \in \mathrm{V_O} \tag{2 10}$$

2 6 3 Stability on Schwartz space

An operator L of the form 2 9 acting on $\mathrm{C}_t^m[\mathcal{S}(\mathrm{R}^n)]^n$ can be written in transformed domain by taking Fourier transform in R^n See theorem A 1 1 If we make following transformation then it can be written as the system of first order equations

$$x_1(t\ \xi) = u(t\ \xi) \tag{2 11}$$

$$x_2(t\ \xi) = \partial_t x_1(t\ \xi) \tag{2 12}$$

$$x_m(t,\xi) = \partial_t x_{m-1}(t\ \xi) \tag{2 13}$$

And denoting

$$X(t\ \xi) = [x_1(t\ \xi), \quad x_m(t\ \xi)]^T \in (\mathrm{C}_t[\mathcal{S}(\mathrm{R}^n)])^m \tag{2 14}$$

$$F(t\ \xi) = [0 \quad , f(t\ \xi)]^T \in (\mathrm{C}_t[\mathcal{S}(\mathrm{R}^n)])^m \tag{2 15}$$

$$X'(t\ \xi) = [\partial_t x_1(t\ \xi) \quad \partial_t x_m(t\ \xi)]^T \tag{2 16}$$

So above equation [redacted] could be written as using above transformation

$$X'(t\ \xi) = A(\xi)X(t\ \xi) + F(t\ \xi) \tag{2 17}$$

Here $A(\xi) \in [\mathrm{C}^\infty(\mathrm{R_n})]^{m\times m}$ has following structure

$$A(\xi) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & & & \\ 0 & 0 & 0 & 1 \\ -a_m(\xi) & -a_{m-1}(\xi) & -a_{m-2}(\xi) & -a_1(\xi) \end{pmatrix} \in [\mathrm{C}^\infty(\mathrm{R_n})]^{m\times m} \tag{2 18}$$

Let $L(\partial_t, \partial_x)$ be a matrix operator Every operator of the form 2 9 can be written in this form as shown in equation B 12 The operator $L(\partial_t, \partial_x)$ has following form

$$L(\partial_t, \partial_x) = I\partial_t + A(\partial_x) \tag{2 19}$$

Here I is an identity matrix and $A(\partial_x)$ is $m \times m$ matrix with entries in C^∞ operators in $x \in R^n$

Now consider the following problem Since all closed loop systems in feedback intercon nection could be looked as a system of partial differential equations so it is equivalent to solving a stabilization problem

$$L(\partial_t\ \partial_x)u(t\ x) = f(t\ x) \tag{2 20}$$

Equation 2 20 shows the action of L on u which belongs to $(C_t[\mathcal{S}(R^n)])^n$ and f also belongs to same space Theorem A 1 1 converts problem 2 20 into ordinary differential equation problem with a parameter in R^nas given below

$$L(\partial_t\ \xi)u(t\ \xi) = f(t\ \xi) \qquad \xi \in R_n \tag{2 21}$$

Corresponding operator in transformed domain is as given below

$$L(\partial_t\ \xi) = I\partial_t + A(\xi) \tag{2 22}$$

We have following result for stability of equation 2 21

Theorem 2 6 1 **If $\lambda_i(\xi) \in \sigma[A(\xi)]$ satisfy following condition for $i = 1, \quad ,n$ and for all $\xi \in R_n$ then system of equations 2 21 is BIBO stable**

$$0 < Im[\lambda_i(\xi)] < \infty \ \ \forall \xi \neq 0$$

Proof Theorem B 3 1 gives condition under which equation 2 21 is well posed If $Im[\lambda\ (\xi)] < \infty$ then equation 2 21 has a unique solution continuous in initial condition and forcing function (see definition 2 5 1) Equation 2 20 is non homogeneous ordinary differential equation in time with parameters in R^n See [16 chapter 3] Now $\xi = \xi_o$ such that $\xi_o \neq 0$

$$[I\partial_t + A(\xi_o)]u(t\ \xi_o) = f(t\ \xi_o) \tag{2 23}$$

Above equation 2 23 is an ordinary differential equation in time so riemanian function for this differential equation exists See [19] for further details This riemanian function s asymptotic behavior is governed by eigenvalues of $A(\xi_o)$

$$\partial_t u(t\ \xi_o) = -A(\xi_o) + f(t,\xi_o) \tag{2 24}$$

$$\frac{\partial u(t,\xi_o)}{\partial t} = -iA(\xi_o) + if(t,\xi\ ) \tag{2 25}$$

Since the equation is well posed for all $\xi \neq 0$ hence for each $\xi = \xi_o$ we can find the corresponding riemanian function $\phi(t\ t\ )$ We can write $\phi_\xi(t\ t_o)$ as a riemanian function for equation 2 21 If the given condition is satisfied then riemaniannian function approaches zero for large t (see [19] [16] and equation 2 25) General solution of equation 2 21 is given below [19]

$$u(t\ \xi) = \int_t^t \phi_\xi(t\ t_1)\, f(t_1\ \xi)\, dt_1 \tag{2 26}$$

For $Im\lambda\,(\xi) < \infty$ then $u(t\ \xi) \in \mathcal{S}(\mathrm{R}\ )$ for all t and if $Im\,\lambda\,(\xi) > 0$ then we have from equation 2 26

$$\begin{aligned}
\|u(t\ \xi)\|_{\alpha\,\beta\,p\,q} &= \|\int_t^t \phi_\xi(t\ t_1)\, f(t_1\ \xi)\, dt_1\|_{\alpha\,\beta\,p\,q} && (2\ 27)\\
&\leq \int_t^t \|\phi_\xi(t\ t_1)\|_{m\ x}\ \|f(t\ \xi)\|_{\alpha\,\beta\,p\,q} && (2\ 28)\\
&\leq M\|\phi_\xi(t\ t_1)\|_{m\ x}\, as\ f(t\ \xi)\ is\ p-bounded && (2\ 29)\\
&\leq MK_1(t)\ \text{[illegible]}\ 0 && (2\ 30)
\end{aligned}$$

We have equation 2 30 as $\phi(t\ t_1)$ is decreasent map with the choice of λ as in condition Hence

$$\|u(t\ \xi)\|_{\alpha\,\beta\,p\,q} \leq \infty\ \forall t \tag{2 31}$$

$\|u(t\ \xi)\|_{\alpha\,\beta\,p\,q}$ is finite so long as $\|f(t\ \xi)\|_{\alpha\,\beta\,p\,q}$ is finite Hence system in equation 2 21 is p stable Also $K_1(t) \to 0$ as $t \to \infty$ Hence also asymptotically stable □

This proves the theorem 2 6 1 Using this result we can prove result for linear time invariant closed loop system

Corollory 2 6 1 Closed loop system modelled by system of ordinary differential equations $x = Ax + f(t)$ is stable if real part of eigenvalues of constant real matrix A is in left half plane

Proof Operator in equation 2 21 acting on x can then be written as

$$x = -\imath A + \imath f(t) \tag{2 32}$$

Using the condition of previous theorem we get $0 < -Re(\lambda\ ) < \infty$ This condition is equivalent to $Re(\lambda_\imath) < 0$ where $\lambda_\imath \in \sigma(A)$ □

Chapter 3

Stabilization Problem over Topological spaces

The concept of topological stability can be used to pose stabilization problem over topological spaces Stabilization problem posed will be more general as the spaces of inputs and outputs considered are more general In this chapter stabilization problem is posed over topological vector spaces This problem has been solved for first order systems over Schwartz space using the condition of stability obtained in the previous chapter

3 1 Stabilization Problem

As introduced earlier T_1 and T_2 will represent topological spaces P and C will respectively represent the Plant system and the Controller system V_{O1} and V_{O2} be the corresponding subspaces of stable inputs and stable outputs (see chapter 2) The controller is connected in feedback connection as shown in figure 1 2 The controller is selected so that the closed loop operator is topologically stable

Definition 3 1 1 [Stabilization Problem] Given a plant P find the class of controllers C such that the closed loop operator T_c has

1 $\mathrm{Image}(T_c^{-1}) \supseteq V_{O1} \times V_{O2}$

2 T_c is one one and continuous

3 $T^{-1}(V_{O2} \times V_{O1}) \subseteq V_{O1} \times V_{O2}$

If the topological spaces are $C(R^+)$ and space of stable inputs and stable outputs are $L_2(R^+)$, then the stabilization problem is the one described in the introduction

3 2 Stabilization problem over Schwartz space

The topological spaces (input and output space) are Schwartz spaces in R^n denoted as $\mathcal{S}(\mathrm{R}^n)$ The spaces of stable inputs and stable outputs are $f(t) \in \mathcal{S}(\mathrm{R}^n)$ such that $\|f(t)\|_{\alpha\,\beta\,p\,q} \rightarrow 0$ as $t \rightarrow \infty$

In this section we will be analyzing first order systems (see equation 2 9) representing plant and controller both In controller design in control theory problem we try to find the class of controllers whose order is less than or equal to that of plant So the controller will either be a constant controller or a first order controller for stabilizing the plant The most general plant from the class in equation 2 9 is

$$\mathrm{P}(\frac{\partial}{\partial t}) = \frac{\partial}{\partial t} + a(\frac{\partial}{\partial t}) \qquad (3\ 1)$$

Similarly the possible controllers are

$$\mathrm{C}_1(\frac{\partial}{\partial x}) = c_1(\frac{\partial}{\partial x}) \qquad (3\ 2)$$

$$\mathrm{C}_2(\frac{\partial}{\partial t}) = \frac{\partial}{\partial t} + c_2(\frac{\partial}{\partial x}) \qquad (3\ 3)$$

These plant and controller operators are written in different form (eq 2 9) as time partial derivatives are not defined as $\partial_t = \frac{1}{\;}\frac{\partial}{\partial t}$ Also the partial derivative in space variable is also not compatible with equation 2 9

3 2 1 Well-posedness of closed loop system

Plant system is of first order i e $m = 1$ (see equation 2 9) Plant operator then can be written as

$$P(\frac{\partial}{\partial x}\ \frac{\partial}{\partial t}) = \frac{\partial}{\partial t} + a(\frac{\partial}{\partial x}) \quad x \in \mathrm{R}^n \quad t \in \mathrm{S} \qquad (3\ 4)$$

As Controllers of order less than or equal to that of plant are of interest in Control Theory Hence possible Controllers for the given Plant operator in equation 3 4 are given below

$$C_1(\frac{\partial}{\partial x}) = b_1(\frac{\partial}{\partial x}) \qquad (3\ 5)$$

$$C_2(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}) = \frac{\partial}{\partial t} + b_2(\frac{\partial}{\partial x}) \qquad (3\ 6)$$

where $a(\ ), b_1(\ ), b_2(\ ) \in \mathrm{C}^\infty(\ )$ Closed loop system is then described by following equations

$$P(\frac{\partial}{\partial t}\ \frac{\partial}{\partial t})u(t,x) + u\ (t,x) = v_1(t,x) \qquad (3\ 7)$$

$$u(t,x) - C_i(\frac{\partial}{\partial t}, \frac{\partial}{\partial x})u_c(t,x) = v_2(t,x) \qquad (3\ 8)$$

Taking Fourier transform in R^n(see equation A 7)

$$P(\frac{\partial}{\partial t}\ i\xi)u(t\ \xi) + u_c(t\ \xi) = v_1(t\ \xi) \qquad (3\ 9)$$

$$u(t\ \xi) - C_i(\frac{\partial}{\partial t}\ i\xi)u_c(t,\xi) = v_2(t\ \xi) \qquad (3\ 10)$$

3 2 2 Case 1 Constant Controller

When $C = C_1$ in equation 3 10 If we substitute equation 3 5 in equation 3 10 we have

$$u(t\ \xi) = v_2(t\ \xi) + b_1(i\xi)u\ (t\ \xi) \qquad (3\ 11)$$

Substituting transformed plant operator in equation 3 9 will give linear differential equation in time

$$\frac{\partial u(t\ x)}{\partial t} + a(i\xi)u(t\ \xi) + u_c(t\ \xi) = u_1(t\ \xi) \qquad (3\ 12)$$

Substituting equation 3 11 in equation 3 12

$$\frac{\partial v_2}{\partial t} + b_1(i\xi)\frac{\partial u_c}{\partial t} + a(i\xi)v_2 + a(i\xi)b_1(i\xi)u_c + u_c = v_1 \qquad (3\ 13)$$

Let us assume that $\frac{\partial v_2}{\partial t} = v_2{}' \in \mathcal{S}(\mathrm{R}^n)\ \forall\ t$ and $a(i\xi)v_2 \in \mathcal{S}(\mathrm{R}^n)$ as $a(i\xi)$ is polynomial in R_n using equation A 6 Now we have if $f = v_1 - av_2 - \frac{\partial v_2}{\partial t}$ then

$$b_1(i\xi)\frac{\partial u_c}{\partial t} + (a(i\xi)b_1(i\xi) + 1)u = f(t\ i\xi) \qquad (3\ 14)$$

Writing in Schwartz format and using theorem B 3 1 we get τ as

$$-Im(\tau) = -Im\left(\frac{i(1 + a(i\xi)b_1(i\xi))}{b_1(i\xi)}\right) < \infty\forall\ \xi \neq 0 \in \mathrm{R}_n \qquad (3\ 15)$$

Hence following proposition can be made

Proposition 3 2 1 Closed loop system in equations 3 9 3 10 is well posed for plant operator as in equation 3 4 and controller operator as in equation 3 5 iff equation 3 15 is satisfied

3 2 3 Case 2 First order Controller

When C_i in equation 3 10 is that of equation 3 6 Plant is same as in CASE 1 The closed loop system is written in following form after taking Fourier transform in R^n(see equation A 7)

Substituting in equations 3 9 3 10

$$\frac{\partial u}{\partial t} + a(i\xi)u + u_c = v_1 \qquad (3\ 16)$$

$$u - \frac{\partial u_c}{\partial t} - b_2(i\xi)u_c = i_2 \qquad (3\ 17)$$

Writing in matrix form

$$\partial_t \begin{bmatrix} u \\ u_c \end{bmatrix} = \begin{bmatrix} -ia(i\xi) & -i \\ -i & b_2(i\xi) \end{bmatrix} \begin{bmatrix} u \\ u_c \end{bmatrix} + \begin{bmatrix} -iv_1 \\ iv_2 \end{bmatrix} \qquad (3\ 18)$$

Therefore $A(\xi)$ will be as given below

$$A(\xi) = \begin{pmatrix} -ia(i\xi) & -i \\ -i & ib_2(i\xi) \end{pmatrix} \qquad (3\ 19)$$

Eigenvalues of $A(\xi)$ are roots of charecteristics equation

$$|\lambda I - A(\xi)| = 0 \qquad (3\ 20)$$

Therefore eigenvalues are

$$\tau_{\pm}(\xi) = \frac{(i[a(i\xi) + b_2(i\xi)] \pm \sqrt{4 - (a(i\xi) - b_2(i\xi))^2})}{2} \qquad (3\ 21)$$

Hence we have following result

Proposition 3 2 2 **If $-Im(\tau_{\pm})$ are bounded above (see theorem B 3 1) $\forall\ \xi \in \mathcal{R}_n$ then system of equations 3 16 3 17 is well posed**

Special Cases

(i) $a(i\xi) = b_2(i\xi) = i\xi$ In such a case $\tau_{\pm} = -1 \pm \sqrt{2}$ (real and distinct) hence well posed using corollary B 3 1

(ii) $a(i\xi) = -b_2(i\xi) = i\xi$ In such a case $\tau_{\pm} = 0\ 5 \pm \sqrt{4\xi^2 + 4}$ (real and distinct) and hence well posed Note in both cases plant and controller are separately well posed

3 3 Stability of closed-loop system

As shown in CASE 1 the requirement for wellposedness for plant 3 4 using controller 3 5 of closed loop system is given by equation 3 15

$$\tau = \frac{i}{b_1(i\xi)} + a(i\xi) \qquad (3\ 22)$$

Using theorem 2 6 1 we have

$$0 < Im\left(\frac{i}{b_1(i\xi)} + a(i\xi)\right) < \infty \tag{3 23}$$

Proposition 3 3 1 Closed loop systems of equations 3 9, 3 10 are stable if there exist a b( ), for a given a( ) such that equation 3 23 is satisfied

If $a(i\xi)$ is real then any positive constant feedback will stabilize plant

Example 3 3 1 Plant $P(\frac{\partial}{\partial t}\ \frac{\partial}{\partial x}) = \frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}$ where $x \in R$ and $t \in S$ Taking fourier transform in R

$$P(\frac{\partial}{\partial t}\ \xi) = \frac{\partial}{\partial t} - \xi^2 \tag{3 24}$$

Any positive constant feedback will stabilize plant

And when $a(i\xi)$ is imaginary then controller 3 5 cannot stabilize the plant 3 4

Example 3 3 2 Plant $P(\frac{\partial}{\partial t}\ \frac{\partial}{\partial x}) = \frac{\partial}{\partial t} + \frac{\partial^3}{\partial x^3}$ where $x \in$ R and $t \in$S Taking fourier transform in R

$$P(\frac{\partial}{\partial t}\ \xi) = \frac{\partial}{\partial t} - i\xi^3 \tag{3 25}$$

Using condition 3 23 we can see that there exists no controller in class of PDE s shown in equation 2 9 of the form 3 5 to stabilize the closed loop system

Similarly for controller 3 6 we can find controller using condition 3 21

Chapter 4

Scope for future work

In this thesis an attempt has made to study the Stabilization Problem over more abstract spaces as Frechet space and topological spaces As we switch to less structured spaces we don't have the concept of magnitude or norm The definition of stability available in literature uses the concept of magnitude which is not available on topological spaces so the notion of stability is generalized on the lines of BIBO stability and it is stronger than V stability

This notion is equivalent to various notions of stability when we suitably structure the topological spaces It has been shown that this notion is related to index of the system operator if it of Fredhlom type over finite dimensional topological spaces

A sufficiency condition is obtained for closed loop stability over $C_t[\mathcal{S}(\mathrm{R}^n)]$ when $f(t,x)$ tends to zero in norm Using this controller has been found for first order systems

4 1 Conclusion

The new notion of stability, on the lines of BIBO stability poses stabilization problem as a invariant subspace problem This stabilization problem has relation with index of the closed loop operator for Fredholm kind of operators This problem if solved will almost completely solve the stabilization problem in Control Theory

4 2 Future work

The inputs and outputs are assumed to be belonging to same linear topological vector space of very well behaved function, Schwartz space Aim would be to attempt the same problem for more general class of linear partial differential operators assuming inputs and outputs belonging to spaces of engineering importance like L_2 The problem of close loop topological stability can also be a part of future work

This problem is essentially an open problem Williams [28] and Desoer [7] [9] have solved this problem for special cases in transformed domain In 1993 some papers were published in SIAM J on Control and Optimization In these works this problem has been as an operator theoretic problem [3] [11] Sule [18] gave a class of controllers in C^n when inputs and outputs are in a ring In time domain this has not been solved completely Objective could be to attempt this problem in time domain

4 3 Methodology

Distribution Theory is a mathematical tool used for solving Linear partial differential equations Stabilization problem for partial differential operators can be tackled if one can solve coupled system of linear partial differential equations in a specified space of functions Distribution Theory gives the distributions satisfying the given linear partial differential equation The biggest advantage of using this theory is that a distribution is always differentiable irrespective of it s continuity If, at all, a classically continuous solution of the linear partial differential equation exists the distributive solution will be continuous in classical sense This approach can be taken if inputs and outputs belong to sufficiently structured function space If one takes the problem of topological stability over linear topological vector space then a geometric approach should be taken Index Theory [1] would be a more appropriate tool to work with less structured function space Index calculation for an operator is purely a geometric problem Atiya Singer s work could be of use in solving this problem atleast for elliptic operators on manifold

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Appendix A

Mathematical Review

A brief review of mathematical concepts used

A 1 Schwartz Space

Schwartz introduced following notations to shorten the expression in the theory of partial differential equations

$$
\begin{aligned}
x &= (x_1 \quad x_n) \;\; x \in \mathrm{R}^n && \text{(A 1)}\\
\partial_x^\alpha &= \partial_{x_1}^{\alpha_1} \quad \partial_{x_n}^{\alpha_n} && \text{(A 2)}\\
\partial_j^k &= (1/i^k)\partial^k/\partial x_j^k && \text{(A 3)}\\
x^\alpha &= x_1^{\alpha_1} \quad x_n^{\alpha_n} && \text{(A 4)}\\
|\alpha| &= \alpha_1 + \quad + \alpha_n && \text{(A 5)}
\end{aligned}
$$

Let X be an open subset of R^n Then $\mathrm{C}_\mathrm{o}^\infty(\mathrm{X})$ denotes all infinitely many times continuously differentiable functions which have compact support in X In the Schwartz s notation this is denoted as $\mathcal{D}(\mathrm{X})$ The dual of $\mathcal{D}(\mathrm{X})$ is the set of all continuous linear functionals on $\mathcal{D}(\mathrm{X})$, denoted as $\mathcal{D}'(\mathrm{X})$

Let us consider the space of C^∞ functions such that equation A 6 is satisfied over $\mathrm{X} \subset \mathrm{R}^n$

$$\|f\|_{\alpha\,\beta} = \sup_{x \in \mathrm{X}} |x^\alpha \partial^\beta f(x)| < \infty \quad \forall\ \alpha\ \beta \qquad \text{(A 6)}$$

This space of functions is called **Schwartz Space**, denoted as $\mathcal{S}(\mathrm{X})$ The dual of Schwartz Space is the space of tempered distributions denoted as $\mathcal{S}'(\mathrm{X})$

Definition A 1 1 [Schwartz] Schwartz space is the set of all functions $f \in \mathrm{C}^\infty(\mathrm{X})$ such that $\|f\|_{\alpha\,\beta}$ is finite for all α and β as in Equation A 6

Fourier Transform in R^nover Schwartz space is defined below

Definition A 1 2 The Fourier transform for $f(x) \in \mathcal{S}(\mathrm{X})$ denoted as $\mathcal{F}(f(x))$ is defined as

$$\mathcal{F}[f(x)] = \int_{\mathrm{X}} f(x) \exp^{-i<x\,\xi>} \, dx \qquad \text{(A 7)}$$

Fourier transform has a remarkable property on Schwartz space which is stated in the form of following theorem

Theorem A 1 1 **Fourier Transform is an isomorphism on Schwartz space of functions**

See [12 chapter 7] for proof of this theorem Inverse Fourier transform is defined in the following way as theorem A 1 1 justifies it s definition

Definition A 1 3 Inverse Fourier transform of $f(\xi)$ is defined as

$$\mathcal{F}^{-1}[f(\xi)] = \int f(\xi) \exp^{i<x\,\xi>} \, d\xi \qquad \text{(A 8)}$$

The dual of $\mathcal{S}(\mathrm{X})$ also has important property and that is why it is easy to work in these spaces for solving partial differential equations

Theorem A 1 2 **Fourier transform is an topological isomorphism on the space of tempered distributions**

This is an extension of theorem A 1 1 by continuity See [12] for proof There is also a topological relation between $\mathcal{S}(\mathrm{X})$ and $\mathcal{S}'(\mathrm{X})$

A 2 Distribution Theory

In previous section, an introduction of distributions is given The formal definition of Distribution is given and few theorems used later are stated

Definition A 2 1 A distribution u in X is a linear form on $\mathrm{C}_0^\infty(\mathrm{X})$ such that for every compact set $\mathrm{K} \subset \mathrm{X}$ there exist constants C and k such that

$$|u(\phi)| \leq C \sum_{|\alpha| \leq k} \sup |\partial^\alpha \phi|, \qquad \phi \in \mathrm{C}_0^\infty(\mathrm{K}) \qquad \text{(A 9)}$$

The set of all distributions in X is denoted by $\mathcal{D}'(X)$. If the same integer k can be used in equation A.9 for every K we say that u is of order $\leq k$. They are denoted as $\mathcal{D}'^k(X)$. Also

$$\mathcal{D}_F'(X) = \cup\mathcal{D}'^k(X) \tag{A.10}$$

The continuity condition in definition A.2.1 gaurantees that u behaves well when acting on functions depending on parameters.

Theorem A.2.1 **If $\phi(x,y) \in C^\infty(X \times Y)$ where Y is an open set in R^m, and if there is a compact set $K \subset X$ such that $\phi(x,y) = 0$ when $x \notin K$, then**

$$y \rightarrow u(\phi(\cdot,y))$$

is a C^∞ function of y if $u \in \mathcal{D}'(X)$, and

$$\partial_y^\alpha u(\phi(\cdot,y)) = u(\partial_y^\alpha \phi(\cdot,y)) \tag{A.11}$$

Proof For fixed $y \in Y$ we have by Taylor's formula

$$\phi(x,y+h) = \phi(x,y) + \sum h_j \partial\phi(x,y)/\partial y_j + \psi(x,y,h) \tag{A.12}$$

$$\sup_x |\partial_x^\alpha \psi(x,y,h) = O(|h|^2) \quad as \quad h \rightarrow 0 \ \forall\alpha \tag{A.13}$$

Hence

$$u(\phi(\cdot,y+h)) = u(\phi(\cdot,y)) + \sum h_j u(\partial\phi(\cdot,y)/\partial y_j) + O(|h|^2) \tag{A.14}$$

which shows that $y \rightarrow u(\phi(\cdot,y,y))$ is differentiable and that

$$\partial/\partial y_j[u(\phi(\cdot,y))] = u(\partial\phi(\cdot,y)/\partial y_j) \tag{A.15}$$

Iteration of this result proves the theorem. □

We know that by pointwise differentiation we may miss something essential at discontinuity but this is not the case with distributional derivative.

Theorem A.2.2 **If $u \in \mathcal{D}'(X)$ where X is an open interval on R and if $u' = 0$, then u is a constant.**

Corollory A.2.1 If $u \in \mathcal{D}'(X)$, where $X \subset R$, and $u' + au = f \in C(X)$ where $a \in C^\infty(X)$, then $u \in C^1(X)$. Thus $u' + au = f$ in classical sense.

Corollory A 2 2 If $u \in \mathcal{D}'(X)$ $X \subset R$ and if

$$u^{(m)} + a_{m-1}u^{(m-1)} + \quad + a_o u = f \in C(X) \qquad (A\ 16)$$

where the coefficients $a_j \in C^\infty(X)$ then $u \in C^m(X)$ in classical sense

Corollory A 2 3 Let $u \in \mathcal{D}'(Y \times I)$ where Y is an open set in $R^n$and I an open interval on R If $\partial_t u = 0$, then

$$u(\phi) = \int u\ (\phi(\ ,t))\, dt \quad \phi \in C_o^\infty(Y \times I) \qquad (A\ 17)$$

where $u_o \in \mathcal{D}'(Y)$ Thus u is a distribution u_o in x independent of t

Theorem A 2 3 **$\mathcal{S}(X)$ is sequentially dense in $\mathcal{S}'(X)$**

See [17, page 74 76] for proof of this theorem Also we know that $\mathcal{D}(R^n) \subset \mathcal{E}'(R^n) \subset \mathcal{S}'(R^n) \subset \mathcal{D}'(R^n)$ Hence we have following proposition

Lemma A 2 1 $C_o^\infty(X)$ is dense in $\mathcal{S}(X)$

The restriction of tempered distribution to $C_o^\infty(R^n)$ is a distribution in $\mathcal{D}'(R^n)$ $u \in \mathcal{S}'(R^n)$ which vanishes on $C_o^\infty(R^n)$ must also vanish on $S(R^n)$

Proposition A 2 1 **A linear map T $\mathcal{S}(R^n) \mapsto \mathcal{C}$ is continuous iff there exists $n \in N$ and $c \in R$ such that for all $\phi \in \mathcal{S}(R^n)$**

$$|<T,\phi>| \leq c \sum_{|\alpha|\leq n\ |\beta|\leq n} \|x^\alpha \partial^\beta \phi\|_{L^\infty(R\ )} \qquad (A\ 18)$$

Proposition A 2 2 **A distribution $T \in \mathcal{D}'(R^n)$ extends uniquely to an element of $\mathcal{S}'(R^n)$ if and only if there exist $n \in N$ and $c \in \mathcal{C}$ such that equation A 18 holds for all $\phi \in \mathcal{D}(R^n)$**

For further details see [17 appendix,chapter 2] and [12 Chapter 7]

A 3 Solution continuity

In this section a theorem is proved on the continuity of the solutions of the class of PDE in equation 2 9 (see [12, chapter 2,3,4] for further details)

Theorem A 3 1 **For every $f \in \mathcal{D}'(X)$, $X \subset R^n$, one can find $f_\alpha \in C(X)$ such that**

$$f(\phi) = \sum \int f_\alpha \, \partial^\alpha \phi \, dx \qquad \phi \in C_o^\infty(X) \tag{A 19}$$

and the sets supp f_α are locally finite If $f \in \mathcal{D}'_F$ the sum can be taken finite

Proof $E(x) = x_{1+}^m \quad x_{n+}^m/(m!)^n$ is a fundamental solution of $P = (\partial_1 \quad \partial_n)^{m+1}$ in R^n and $E \in C^k$ if $m > k$ If $f \in \mathcal{E}'^k(R^n)$ it follows that $u = E * f \in C(R^n)$ satisfies the equation Pu=f Choosing a partition of unity $1 = \sum \psi_j$ in X Let $\chi_j \in C_o^\infty(X)$ be equal to 1 near the support of ψ_j If m_j is larger than the order of $\chi_j f$ we have just seen that one can find $u_j \in C(R^n)$ such that $(\partial_1 \quad \partial_n)^{m_j+1} u_j = \chi_j f$ It follows that

$$\begin{aligned} < f \ \phi > &= \sum < \chi_j f \ \psi_j \phi > && \text{(A 20)} \\ &= \sum < u_j \ \pm(\partial_1 \quad \partial_n)^{m_j+1}(\chi_j \phi) > && \text{(A 21)} \\ &= \sum \int u_j \partial^j \psi_j \, \phi dx && \text{(A 22)} \\ f(\phi) &= \sum \int f_\alpha \partial^\alpha \phi dx && \text{(A 23)} \end{aligned}$$

Hence proved □

With the help of above theorem we prove following result

Theorem A 3 2 **Let $X = I \times Y \subset R^{n+1}$ where $Y \subset R^n$ and Y is open set and I is an open interval on R, and assume that $u \in \mathcal{D}'(X)$ satisfies a differential equation of the form**

$$\partial_t^m u + a_{m-1} \partial_t^{m-1} u + \quad + a_o u = f \tag{A 24}$$

where a_j is a differential operator in R^n with coefficients in $C^\infty(X)$ and is a continuous function of $t \in I$ with values in $\mathcal{D}'(Y)$ Then it follows that u is a C^m function of $t \in I$ with values in $\mathcal{D}'(Y)$

Proof If we allow u to be vector valued and the coefficients to be square matrices the proof reduces to the case m=1 We assume from now on m=1 Shrinking Y and I if necessary, we can, by theorem A 3 1, write u as finite sum

$$u = \sum_{|\alpha| \le \mu} \partial^\alpha u_\alpha \tag{A 25}$$

where $u_\alpha \in C(X)$ and also

$$f = \sum_{|\alpha|\leq\mu \quad \alpha_t=0} \partial^\alpha f_\alpha \qquad f_\alpha \in C(X) \tag{A 26}$$

If $u_\alpha = 0$ when $\alpha_t > 0$ then u and $\partial_t u = f - a_o u$ are continuous with values in $\mathcal{D}'(Y)$ Otherwise let ν be the smallest integer such that $u_\alpha = 0$ when $\alpha_t > \nu$ Then we have
$\partial_t u = f - a_o u = \sum \partial^\alpha f_\alpha - \sum a_o \partial^\alpha u_\alpha = \sum \partial^\alpha v_\alpha = \partial_t \sum \partial^\alpha V_\alpha$
where v_α and V_α are continuous functions and $v_\alpha = 0$ when $\alpha_t > \nu$ and $V_\alpha = 0$ when $\alpha_t > \nu - 1$ In fact we obtain the expression involving v_α by commuting the coefficients of a_o through differentiation successively a_o only involves x derivatives To pass to the representation with V_α we just take out a factor ∂_t if $\alpha_t > 0$ and take a primitive function of v_α with respect to t if $\alpha_t = 0$ If $w = u - \sum \partial^\alpha V_\alpha$ we have $\partial_t w = 0$ Hence $W = \sum \partial^\alpha w_\alpha(x)$ by corollary A 2 3 and theorem A 3 1 Now we obtain

$$u = \sum \partial^\alpha w_\alpha(x) + \sum \partial^\alpha V_\alpha \tag{A 27}$$

where $\alpha_t \leq \nu - 1$ in the sum Iterating the argument ν times we obtain a similar representation when $\alpha_t = 0$ in every term □

Appendix B

Well-Posedness

Now we look at the following problem

$$P(\partial_t,\partial_x)\,u(t\ x) = f(t\ x) \qquad \text{(B 1)}$$

where $P(\partial_t\ \partial_x)$ is of the form as in equation 2 9 $C_t^k[\mathcal{S}(X)]$ which is the class of functions k times continuously differentiable in t and valued in $\mathcal{S}(X)$ $f(t\ x) \in C_t[\mathcal{S}(R^n)]$

B 1 Hadamard-Perowsky condition

Since $u(t,x)$ is valued in Schwartz space hence we take Fourier transform in $R^n$and work in transformed domain Using theorem A 1 1 and equation A 7 defining Fourier transform we get from equation B 1

$$P(\partial_t,\xi)\,u(t,\xi) = f(t\ \xi) \qquad \text{(B 2)}$$

Above equation B 2 is a ordinary differential equation in time with coefficients C^∞ functions in R^n The characteristic equation of equation B 2 is

$$P(\tau\ \xi) = 0 \text{ where } \xi \in R^n\ \tau \in \mathcal{C} \qquad \text{(B 3)}$$

Definition B 1 1 The differential operator satisfies Hadamard Petrowsky condition for forward evolution iff there exists c, n such that the roots of equation B 3 τ satisfy

$$e^{-Im\tau} \le c<\xi>^n \qquad \text{(B 4)}$$

HP condition, as defined in definition B 1 1, has different forms Taking logarithms on both sided shows that

$$-Im(\tau) \le c' \ln(2+|\xi|) \qquad \text{(B 5)}$$

that is $-Im(\tau)$ can grow at most logarithmically in $|\xi|$ As τ s are roots of an algebraic equation only power law growth is possible [13 appendix] Thus the only way it can be satisfied is when $Im(\tau)$ is bounded above

B 2 Well-posedness of equation

Again going back to equation B 2 which is an ordinary differential equation in t with coefficients in C^∞ in parameter $\xi \in R_n$, dual of R^n Now $f(t\ \xi) \in C_t[\mathcal{S}(R^n)]$ using theorem A 1 1

Theorem B 2 1 **Pu = f, where P is of the form 2 9, is well posed if roots of equation B 3 satisfy HP condition (see definition B 1 1) for forward evolution**

See [17 Chapter 3]

B 3 Well-posedness of system of equations

Now looking at equation B 2 after making following transformation which is always possible for operators of the form as in equation 2 9

$$x_1(t\ \xi) = u(t\ \xi) \tag{B 6}$$

$$x_2(t,\xi) = \partial_t x_1(t\ \xi) \tag{B 7}$$

$$x_m(t\ \xi) = \partial_t x_{m-1}(t\ \xi) \tag{B 8}$$

And denoting

$$X(t,\xi) = [x_1(t\ \xi) \qquad x_m(t\ \xi)]^T \in (C_t[\mathcal{S}(R^n)])^m \tag{B 9}$$

$$F(t,\xi) = [0 \qquad f(t,\xi)]^T \in (C_t[\mathcal{S}(R^n)])^m \tag{B 10}$$

$$X'(t,\xi) = [\partial_t x_1(t,\xi) \qquad ,\partial_t x_m(t\ \xi)]^T \tag{B 11}$$

So equation B 2 could be written as, using above transformation

$$X'(t\ \xi) = A(\xi)X(t\ \xi) + F(t\ \xi) \tag{B 12}$$

Here $A(\xi) \in [\mathrm{C}^\infty(\mathrm{R_n})]^{m\times m}$ has following structure

$$A(\xi) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & & & \\ 0 & 0 & 0 & 1 \\ -a_m(\xi) & -a_{m-1}(\xi) & -a_{m-2}(\xi) & -a_1(\xi) \end{pmatrix} \in [\mathrm{C}^\infty(\mathrm{R_n})]^{m\times m} \qquad \text{(B 13)}$$

It is worth noticing that $\mathrm{P}(\tau, \xi)$ is the characteristic polynomial of $A(\xi)$ If $\lambda_j(\xi)$ denote an element of spectrum of $A(\xi)$ then well posedness is assured if $Im(\lambda_j(\xi))$ is bounded from above for all $j = 1, \quad m$ and for all $\xi \neq 0 \in \mathrm{R}_n$

Hence now sufficient condition for system of equations for well posedness will suffice Plant and Controller interconnected in feedback interconnection is modelled as system of PDE's

Theorem B 3 1 [Hadamard Mizohata] Let us consider system of equations B 12 The Cauchy problem is well posed for this equation iff each element of spectrum of $A(\xi)$ has bounded imaginary part for all $t \in \mathrm{S}$

See [10, Evolution systems] and also see [14] In addition to this result we can give some sufficient conditions which will assure well posedness

Corollory B 3 1 The system of equations B 12 is well posed whenever one of the following conditions are fulfilled

1 For any $\xi \in \mathrm{R}_n$, the matrix $A(\xi)$ is self adjoint

2 For any $\xi \in \mathrm{R}_n$, the matrix has real and distinct elements in spectrum of $A(\xi)$ for $\xi \neq 0$

Theses are sufficient conditions See [17]

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